

SPARK DEFICIENT GABOR FRAMES

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ABSTRACT. The theory of Gabor frames of functions defined on finite abelian groups was initially developed in order to better understand the properties of Gabor frames of functions defined over the reals, however, during the last twenty years it has an interest on its own. One of the fundamental questions asked in this finite setting, is the existence of full spark Gabor frames. The author proved the existence [13], as well as constructed such frames, when the underlying group is finite cyclic. In this paper, we resolve the non-cyclic case; in particular, we show that there can be no full spark Gabor frames of windows defined on finite abelian non-cyclic groups. We also prove that all eigenvectors of certain unitary matrices in the Clifford group in odd dimensions generate spark deficient Gabor frames, as well. Furthermore, similarities between the uncertainty principles concerning the finite dimensional Fourier transform and the short-time Fourier transform are discussed.

1. INTRODUCTION

The Gabor frame of a function $f \in L^2(\mathbb{R})$ is the set of all time–frequency translates of f , that is, the set of all functions of the form $e^{2\pi ixy} f(x-t)$, for $y, t \in \mathbb{R}$, and it is a fundamental concept in time–frequency analysis and frame theory [15]. The function f usually represents a signal, t the time delay, and the pointwise multiplication by $e^{2\pi ixy}$ is the frequency “shift”. Through sampling and periodization [3] one passes to the finite version of a Gabor frame, namely the shift–frequency translates of a complex function defined on a finite cyclic group. Even though finite dimensional Gabor frames were studied in order to analyze the properties of continuous signals, they later developed an interest on their own.

A conjecture by Heil, Ramanathan, and Topiwala from 1996 [7] states that any finite set of a Gabor frame of a nonzero $f \in L^2(\mathbb{R})$ is linearly independent, and it is still open. Here, we will address the discrete version of this conjecture¹, whether there exist full spark Gabor frames, when the function is defined on a finite abelian group, G . As this set consists of $|G|^2$ elements in a $|G|$ -dimensional space, we require that any selection of $|G|$ vectors is linearly independent, which is the definition of the full spark property. This problem has been completely solved by the author when G is cyclic [13].

For non-cyclic groups, it was previously only known that full spark Gabor frames *do not exist* for functions defined on the Klein group, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ [15]. We shall extend this argument to any finite abelian non-cyclic group, in the following way: first, we show that the full spark property is hereditary with respect to the group. Therefore, in order to show that no full spark Gabor frame exists, it suffices to restrict our attention to groups of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, for p odd prime. Thus is proved the first main result of this paper:

Theorem 1.1. *Let G be a finite abelian, non-cyclic group. Then, for any $f \in \mathbb{C}^G$, the Gabor frame generated by f is spark deficient.*

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¹In any case, we fall into either one of the following two extremes: either there are no full spark Gabor frames, or they do exist for almost all functions f , see Proposition 2.3.

We will also revisit the cyclic case and prove that all eigenvectors of Clifford unitaries whose (projective) order is not coprime to the dimension N , for N odd, generate spark deficient Gabor frames, extending some results in [4]. Lastly, we investigate a possible connection between uncertainty principles with respect to the discrete and short-time Fourier transforms.

The paper is organized as follows: in section 2, we will lay down the definitions and the necessary background related to the results of this paper. In section 3, we will prove that full spark Gabor frames do not exist over finite abelian non-cyclic groups. Section 4 revisits the cyclic case, where we find some special vectors that generate spark deficient Gabor frames, and section 5 deals with uncertainty principles.

2. BACKGROUND

2.1. Notation. Throughout this note, G will denote a finite abelian group written additively, and \mathbb{C}^G will denote the set of all complex valued functions defined on G . \mathbb{C}^N is equipped with an inner product $\langle \cdot, \cdot \rangle$, defined as follows:

$$\langle x, y \rangle = \sum_{i=1}^N x_i \bar{y}_i,$$

for $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N)$. Only in section 4.2 will we use the bra-ket notation, $\langle x|y \rangle$, with the caution that complex conjugation is taken on the coordinates of x . We remind that $|x \rangle$ denotes a column vector in \mathbb{C}^N , and $\langle x|$ is its conjugate transpose; hence $|x \rangle \langle x|$ is the 1-dimensional projector onto $|x \rangle$.

Furthermore, we decided to use $\mathbb{Z}/N\mathbb{Z}$ for the ring of residues mod N , and reserve \mathbb{Z}_p for the ring of p -adic integers. Similarly, \mathbb{Q}_p denotes the field of p -adic rational numbers.

For any $f \in \mathbb{C}^N$ denote by \widehat{f} the (unnormalized) Fourier transform of f ; that is, $\widehat{f} = W_N f$, where $W_N = (\omega^{ij})_{i,j=0}^{N-1}$, the character table of $\mathbb{Z}/N\mathbb{Z}$, with $\omega = e^{2\pi i/N}$, and finally, let $\|f\|_0$ denote the cardinality of the support of f .

Two operators U and V on \mathbb{C}^N will be equal up to a phase if $U = e^{i\theta} V$; this will also be denoted as

$$U \doteq V.$$

The *projective* order of an operator U is then defined to be the smallest nonnegative integer m for which $U^m \doteq I$. Finally, the conjugate transpose of U is denoted by U^* .

2.2. Definitions. For any $x \in G$ and $\xi \in \widehat{G}$, we define the operators $T_x, M_\xi : \mathbb{C}^G \longrightarrow \mathbb{C}^G$, with $T_x f(g) = f(g - x)$ and $M_\xi f(g) = \xi(g) f(g)$, for any $f \in \mathbb{C}^G$, $g \in G$. The T_x are called *translation operators*, and the M_ξ *modulation operators*. For any $\lambda = (x, \xi) \in G \times \widehat{G}$ the operators $\pi(\lambda) = M_\xi T_x$ are called *time-frequency shift operators*. We have

$$M_\xi T_x = \xi(x) T_x M_\xi,$$

or, in other words, M_ξ and T_x commute up to a phase. From this fact we get a faithful projective representation

$$\rho : G \times \widehat{G} \longrightarrow \text{PGL}(\mathbb{C}^G),$$

which is also irreducible [5, 15].

For a subset $\Lambda \subseteq G \times \widehat{G}$ and $f \in \mathbb{C}^G \setminus \{0\}$, the set

$$(f, \Lambda) = \{\pi(\lambda) f \mid \lambda \in \Lambda\}$$

is called a *Gabor system*; if it spans \mathbb{C}^G , it is called a *Gabor frame*. This certainly happens when $\Lambda = G \times \widehat{G}$ due to the irreducibility of ρ ; in this case, it is also called a *Weyl-Heisenberg orbit*.

Definition 2.1. A set Φ of M vectors in \mathbb{C}^N is called a *frame* if it spans \mathbb{C}^N . Thus, Theorem 1.1 follows directly from the theorems below. In this case, we must have $M \geq N$. The *spark* of Φ , denoted by $\text{sp}(\Phi)$, is the size of the smallest linearly dependent subset of Φ . When $\text{sp}(\Phi) = N + 1$, we say that Φ is *full spark*, otherwise we call Φ *spark deficient*.

A frame Φ is full spark if and only if every set of N elements of Φ is a basis. Other definitions are also found in literature; for example, in this case we also say that the vectors of Φ are in *general linear position*, or also that Φ possesses the *Haar property* [15].

Definition 2.2. For a window $\varphi \in \mathbb{C}^G$, denote by $V_\varphi : \mathbb{C}^G \rightarrow \mathbb{C}^{G \times \widehat{G}}$ the short-time Fourier transform with window φ , defined by

$$V_\varphi f = (\langle M_\xi T_x \varphi, \bar{f} \rangle)_{(x, \xi) \in G \times \widehat{G}}.$$

2.3. Gabor systems of $|G|$ vectors. An element $f \in \mathbb{C}^G$ will interchangeably be viewed as a vector in \mathbb{C}^N , where $N = |G|$, and as a function $f : G \rightarrow \mathbb{C}$. Then, the Gabor system (f, Λ) with $|\Lambda| = N$ is linearly independent if and only if the determinant of the matrix whose columns consist of the coordinates of the vectors $\pi(\lambda)f$, $\lambda \in \Lambda$, is nonzero. This matrix is denoted by D_Λ , and is well-defined up to permutation of its columns. The determinant is denoted by $P_\Lambda = \det(D_\Lambda)$, and is well-defined up to a sign, so it makes sense to ask whether P_Λ is nonzero or not.

The most important property of P_Λ , however, is the fact that it is a homogeneous polynomial of degree N in N variables, when the coordinates of f are viewed as independent variables. So, the existence of an element f such that (f, Λ) is linearly independent happens precisely when P_Λ is a nonzero polynomial. Investigating the properties of these polynomials P_Λ sheds light on the existence of Gabor frames in general linear position.

A first crucial observation regarding linear independence, comes from the following:

Proposition 2.3. *There is a full spark Gabor frame defined over G , if and only if, for every $\Lambda \subseteq G \times \widehat{G}$ with $|\Lambda| = N$ there is an $f \in \mathbb{C}^G$ such that (f, Λ) is linearly independent. Moreover, either all windows $\varphi \in \mathbb{C}^G$ generate spark deficient Gabor frames, or almost all windows generate full spark Gabor frames.*

Proof. One direction follows from definition; if $(f, G \times \widehat{G})$ is full spark, then obviously every Gabor system (f, Λ) is linearly independent, for $|\Lambda| = N$. On the other hand, if for every $\Lambda \subseteq G \times \widehat{G}$ with $|\Lambda| = N$ there is some $f \in \mathbb{C}^G$ such that (f, Λ) is linearly independent, this means that all such polynomials P_Λ are nonzero. The zero set of every such polynomial is of Lebesgue measure zero, and since they are finitely many, this yields that almost any $f \in \mathbb{C}^G$ avoids the zero set of these polynomials, hence $(f, G \times \widehat{G})$ is full spark.

For the second part, we observe that if at least one of the polynomials P_Λ is zero, then all Gabor frames defined over G are spark deficient. Otherwise, as we have already shown, almost all Gabor frames are full spark. \square

2.4. The Weyl–Heisenberg and Clifford groups. We restrict our attention to cyclic groups $G = \mathbb{Z}/N\mathbb{Z}$ of odd order, for convenience, as the results of this subsection will only be used towards the construction of spark deficient Gabor frames over cyclic groups. The group generated by the translation and modulation operators is

$$\{\omega^k M^b T^a | a, b, k \in \mathbb{Z}/N\mathbb{Z}\},$$

where $\omega = e^{2\pi i/N}$, $T = T_1$ (see 2.2) and M is the operator with the property $Mf(g) = \omega^g f(g)$ for all $g \in \mathbb{Z}/N\mathbb{Z}$ and $f \in \mathbb{C}^N$, and is called the *Weyl–Heisenberg group* of G . Sometimes [1, 4, 20], these representatives over the center are considered:

$$D_\lambda = \tau^{\lambda_1 \lambda_2} T^{\lambda_1} M^{\lambda_2},$$

where $\lambda = (\lambda_1, \lambda_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, $\tau = \omega^{\frac{N+1}{2}}$.

It is known that all irreducible projective representations of $(\mathbb{Z}/N\mathbb{Z})^2$ of dimension N are unitarily equivalent to ρ [19] (see also Proposition 3.2 [5]). The normalizer of the Weyl–Heisenberg group in the group of unitary matrices in N dimensions is called the *Clifford group*, denoted by $C(N)$. The quotient of $C(N)$ by the Weyl–Heisenberg group is isomorphic to $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, hence ρ can be extended to a faithful irreducible projective representation of $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$. Restricting this representation to the right factor, $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, we get a projective representation $F \mapsto U_F$, for $F \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$. The unitary matrices U_F act on the Weyl–Heisenberg group by conjugation:

$$U_F D_\lambda U_F^* = D_{F\lambda}.$$

More precisely, the following is true:

Theorem 2.4 (Theorem 1 [1], N odd). *There exists a unique isomorphism*

$$f : (\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}) \longrightarrow C(N)/I(N)$$

with the property $U D_\lambda U^ = \omega^{[\varphi, F\lambda]} D_{F\lambda}$ for any $U \in f(\varphi, F)$, where $I(N)$ is the center of $C(N)$, and $[\varphi, \chi] = \varphi_2 \chi_1 - \varphi_1 \chi_2$.*

This yields the following theorem:

Theorem 2.5. *For N odd, there is a unique faithful irreducible projective representation of $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ of dimension N , up to unitary equivalence.*

Proof. We denote by ρ the standard representation of $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, and let $\tilde{\rho}$ denote another representation of dimension N with the same properties. By Weyl’s theorem [5, 19], we may assume without loss of generality that

$$\rho|_{(\mathbb{Z}/N\mathbb{Z})^2} = \tilde{\rho}|_{(\mathbb{Z}/N\mathbb{Z})^2}.$$

Since the image of $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ acts by conjugation on the image of $(\mathbb{Z}/N\mathbb{Z})^2$, the image of $\tilde{\rho}$ will also be contained in $C(N)$. According to Theorem 2.4, for any $F \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, $\rho(F)$ and $\tilde{\rho}(F)$ should differ by an element of $\rho((\mathbb{Z}/N\mathbb{Z})^2)$, that is

$$\tilde{\rho}(F) \doteq D_\varphi U_F.$$

We will investigate the possibilities of φ when $F = S$ or T , the generators of $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which satisfy $S^2 = (ST)^3 = -I$, as well as when $F = -I$. Assume therefore, that

$$\tilde{\rho}(T) \doteq D_\chi U_T, \quad \tilde{\rho}(S) \doteq D_\psi U_S, \quad \tilde{\rho}(-I) \doteq D_\mu U_{-I}.$$

Since $\tilde{\rho}(S)^2 \doteq \tilde{\rho}(-I)$, we must have

$$\tilde{\rho}(-I) \doteq D_\mu U_{-I} \doteq (D_\psi U_S)^2 = D_{\psi+S\psi} U_{-I},$$

hence

$$\mu = (I + S)\psi.$$

On the other hand, $\tilde{\rho}(-T) \doteq \tilde{\rho}(-I)\tilde{\rho}(T) \doteq \tilde{\rho}(T)\tilde{\rho}(-I)$, whence

$$D_{\chi+T\mu} U_{-T} \doteq \tilde{\rho}(T)\tilde{\rho}(-I) \doteq \tilde{\rho}(-I)\tilde{\rho}(T) \doteq D_{\mu-\chi} U_{-T},$$

therefore

$$2\chi = (I - T)\mu,$$

thus

$$2\chi = (I - T)(I + S)\psi.$$

Now, let

$$\lambda = -(I - S)^{-1}\psi = -2^{-1}(I + S)\psi,$$

so that

$$\chi = -(I - T)\lambda, \quad \psi = -(I - S)\lambda.$$

Then,

$$D_\lambda(D_\chi U_T)D_\lambda^* \doteq D_{\lambda+\chi-T\lambda}U_T = U_T$$

and

$$D_\lambda(D_\psi U_S)D_\lambda^* \doteq D_{\lambda+\psi-S\lambda}U_S = U_S,$$

while obviously $D_\lambda D_\varphi D_\lambda^* \doteq D_\varphi$, thus proving that

$$D_\lambda \tilde{\rho}(\varphi, F) D_\lambda^* \doteq \rho(\varphi, F),$$

for all $(\varphi, F) \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z}) \rtimes (\mathbb{Z}/N\mathbb{Z})^2$, or in other words, ρ and $\tilde{\rho}$ are unitarily equivalent, completing the proof. \square

Another way to obtain such a representation is the following: let

$$N = p_1^{r_1} \cdots p_s^{r_s}$$

be the prime factorization of N . By Chinese Remainder Theorem we obtain

$$(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \text{SL}(2, \mathbb{Z}/N\mathbb{Z}) \cong \prod_{i=1}^s (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^2 \rtimes \text{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z}),$$

and let

$$\text{SL}(2, \mathbb{Z}/N\mathbb{Z}) \ni F \mapsto (F_i)_{1 \leq i \leq s} \in \prod_{i=1}^s \text{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$$

be the natural map according to the isomorphism above; that is, F_i is the matrix obtained by F reducing its entries mod $p_i^{r_i}$. Assuming that $V_i \cong \mathbb{C}^{p_i^{r_i}}$ is the faithful irreducible projective representation of $(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^2 \rtimes \text{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$ constructed as above, then we also see that $V_1 \otimes V_2 \otimes \cdots \otimes V_s$ is also a faithful irreducible representation of $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ (Theorem 10 [16]), and hence unitarily equivalent to the standard one. This shows that U_F is, up to unitary equivalence, equal to the Kronecker product of the U_{F_i} , thus

$$(2.1) \quad \text{Tr } U_F = \prod_{i=1}^s \text{Tr } U_{F_i},$$

a fact also pointed out in [4].

3. GABOR FRAMES OVER NON-CYCLIC GROUPS

First we show that the full spark property is hereditary.

Lemma 3.1. *Let G be a finite abelian group and H a subgroup, such that no windows defined on H generate full spark Gabor frames. Then, there exist no windows defined on G that generate full spark Gabor frames.*

Proof. By hypothesis, there exists a set of pairs $(h_i, \xi_i) \in H \times \widehat{H}$, $1 \leq i \leq |H|$, such that the vectors $M_{\xi_i} T_{h_i} \varphi$ are linearly dependent for any choice of $\varphi \in \mathbb{C}^H$. Now, extend the characters ξ_i to G in all possible ways. In this way, we obtain pairs in $G \times \widehat{G}$ of the form (h, ξ) , where $h = h_i$ and $\xi|_H = \xi_i$, for some i ; the number of these pairs is exactly $|G|$, as there are $|G/H|$ ways to extend a character of H to a character of G .

Next, consider an arbitrary window $\psi \in \mathbb{C}^G$. Since the vectors $M_{\xi_i} T_{h_i} \psi|_H$ are linearly dependent on \mathbb{C}^H , there is a nonzero vector $f \in \mathbb{C}^H$ such that all inner products $\langle M_{\xi_i} T_{h_i} \psi_H, \bar{f} \rangle = 0$.

Denote by F the unique window of \mathbb{C}^G for which we have $F|_H = f$ and $\text{supp}(F) \subseteq H$ (also a nonzero window). Then, for all i and $\xi \in \widehat{G}$ with $\xi|_H = \xi_i$ we have

$$\langle M_\xi T_{h_i} \psi, \bar{F} \rangle = \sum_{g \in G} \xi(g) \psi(g - h_i) F(g) = \sum_{h \in H} \xi_i(h) \psi(h - h_i) f(h) = \langle M_{\xi_i} T_{h_i} \psi_H, \bar{f} \rangle = 0,$$

which shows that these $|G|$ pairs $(h, \xi) \in G \times \widehat{G}$ always give linearly dependent vectors, as desired. \square

Since we wish to prove that there exist no windows over any finite abelian non-cyclic group that generate full spark Gabor frames, it suffices to do so for groups of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, for p prime, due to the fundamental theorem of finite abelian groups; if such a group is non-cyclic, then it must have a subgroup of this form. Thus, Theorem 1.1 follows directly Theorem 3.3.

When $p = 2$, this was already proven, therefore by Lemma 3.1 we know that any window defined on a group containing a copy of the Klein group as a subgroup cannot generate a full spark Gabor frame. We provide an alternative proof of this statement, more in line with the proof of Lemma 3.1, which also gives us an estimate on the minimum value of $\|V_\varphi f\|_0$.

Theorem 3.2. *Let G be a finite abelian group that has a subgroup isomorphic to the Klein 4-group. Then, there are no Gabor frames $(f, G \times \widehat{G})$ in general linear position; furthermore, we have*

$$\min \|V_\varphi f\|_0 \leq N^2 - 3N/2.$$

Proof. Let K be the subgroup of G isomorphic to the Klein 4-group. For $f \in \mathbb{C}^G$ define \bar{f} satisfying $\bar{f}(g) = \overline{f(g)}$ for all $g \in G$, and define on \mathbb{C}^G an inner product given by

$$\langle f, h \rangle = \sum_{g \in G} f(g) \bar{h}(g).$$

By standard character theory, there are three nontrivial characters on K , and each one of them extends to $N/4$ characters on G , where $N = |G|$. In total, there are $3N/4$ characters on G whose restriction on K is nontrivial.

Let ξ be such a character, and let $f \in \mathbb{C}^G \setminus \{0\}$ be arbitrary. Let $a \in K$ be such that $\xi(a) = -1$; there are two such elements of K , and so we consider the Gabor system consisting of time-frequency translates of the form

$$M_\xi T_a f, \quad \xi \text{ nontrivial on } K, a \in K \text{ with } \xi(a) = -1.$$

This system has $3N/2 > N$ elements; we will show that each one of them is orthogonal to \bar{f} , and therefore the full Gabor frame $(f, G \times \widehat{G})$ cannot be in general linear position. Indeed,

$$\begin{aligned} \langle M_\xi T_a f, \bar{f} \rangle &= \sum_{g \in G} \xi(g) f(g - a) \bar{f}(g) \\ &= \sum_{g \in G} \xi(g + a) f(g) \bar{f}(g + a) \\ &= \sum_{g \in G} \xi(g) \xi(a) f(g - a) \bar{f}(g) \\ &= - \sum_{g \in G} \xi(g) f(g - a) \bar{f}(g) \\ &= - \langle M_\xi T_a f, \bar{f} \rangle, \end{aligned}$$

so $\langle M_\xi T_a f, \bar{f} \rangle = 0$. This also shows that $\|V_f \bar{f}\|_0 \leq N^2 - 3N/2$, proving the second part of the Theorem. \square

Theorem 3.3. *There are no full spark Gabor frames over $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, for p prime.*

Proof. The case $p = 2$ has already been proven, so we may assume that p is odd. As in the previous two proofs, we consider an arbitrary window $z \in \mathbb{C}^G$, and then try to find a nonzero vector that is orthogonal to at least $|G| = p^2$ shift–frequency translates of z . In order to find this desirable set of translates, we arrange the coordinates of z in an array; here, we identify $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ with the finite field \mathbb{F}_q , $q = p^2$, and $\theta \in \mathbb{F}_q \setminus \mathbb{F}_p$:

$$(3.1) \quad \begin{array}{|c|c|c|c|} \hline z_0 & z_\theta & \cdots & z_{-\theta} \\ \hline z_1 & z_{\theta+1} & \cdots & z_{-\theta+1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline z_{-1} & z_{\theta-1} & \cdots & z_{-\theta-1} \\ \hline \end{array}.$$

We denote this $p \times p$ matrix by Z . The column vectors in $\mathbb{C}^{\mathbb{F}_p}$ from left to right they are denoted by $Z_0, Z_\theta, \dots, Z_{-\theta}$, respectively, and similarly, the row vectors by $Z'_0, Z'_1, \dots, Z'_{p-1}$. Next, consider the vector $x \in \mathbb{C}^{\mathbb{F}_q}$ whose matrix representation is precisely $X = (\text{adj } Z)^*$, where $\text{adj } Z$ denotes the *adjugate* matrix of Z ; we denote its columns by $X_0, X_\theta, \dots, X_{-\theta}$ and its rows by $X'_0, X'_1, \dots, X'_{p-1}$. The vector x could be zero, however this happens for a set of Lebesgue measure zero. In particular, x is zero precisely when all the $(p-1) \times (p-1)$ minors of Z are zero, but all of them are nonzero polynomials on the coordinates of z , which shows that for almost all choices of z , x is nonzero. If we prove that the Gabor frames with windows z possessing that property are spark deficient, then by Proposition 2.3 we get that all Gabor frames over $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ are spark deficient.

We have

$$\det Z \cdot I = Z^T \bar{X} = Z \bar{X}^T.$$

The (a, b) entry of $Z^T \bar{X}$ is $\langle Z_{a\theta}, X_{b\theta} \rangle$, and similarly for $Z \bar{X}^T$ is $\langle Z'_a, X'_b \rangle$. We thus obtain

$$(3.2) \quad \langle Z_{a\theta}, X_{b\theta} \rangle = \langle Z'_a, X'_b \rangle = \delta_{ab} \det Z,$$

for every $a, b \in \mathbb{F}_p$, where δ_{ab} is the usual Kronecker delta. Then, for every $a \in \theta\mathbb{F}_p^*$ and $\xi \in \widehat{\mathbb{F}}_q$ with $\xi|_{\mathbb{F}_p} = \mathbf{1}_{\mathbb{F}_p}$, we get due to (3.2)

$$\langle M_\xi T_a z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b\theta) \langle Z_{b\theta-a}, X_{b\theta} \rangle = 0.$$

This number of shift–frequency translates is $p(p-1)$, and we have just established that x is orthogonal to all of them. Furthermore, if $a \in \mathbb{F}_p^*$ and $\xi \in \widehat{\mathbb{F}}_q$ with $\xi|_{\mathbb{F}_p} = \mathbf{1}_{\mathbb{F}_p}$, we also get due to (3.2)

$$\langle M_\xi T_a z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b) \langle Z'_{b-a}, X'_b \rangle = 0.$$

So far we have $2p(p-1) > p^2$ translates of z orthogonal to x , so this already takes care of the spark deficiency of any Gabor frame over G . We will find more translates orthogonal to x ; let's put $a = 0$ and $\xi \in \widehat{\mathbb{F}}_q$ with $\xi|_{\mathbb{F}_p} = \mathbf{1}_{\mathbb{F}_p}$, but $\xi \neq \mathbf{1}_{\mathbb{F}_q}$. Then, again we have by (3.2)

$$\langle M_\xi z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b\theta) \langle Z_{b\theta}, X_{b\theta} \rangle = \det Z \sum_{b \in \mathbb{F}_p} \xi(b\theta) = 0,$$

since $\xi|_{\theta\mathbb{F}_p} \neq \mathbf{1}_{\theta\mathbb{F}_p}$. This number of pairs is exactly $p-1$.

Next, we still consider $a = 0$, but $\xi \in \widehat{\mathbb{F}}_q$ satisfies with $\xi|_{\mathbb{F}_p} \neq \mathbf{1}_{\mathbb{F}_p}$ and $\xi|_{\theta\mathbb{F}_p} = \mathbf{1}_{\theta\mathbb{F}_p}$. Then,

$$\langle M_\xi z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b) \langle Z'_b, X'_b \rangle = \det Z \sum_{b \in \mathbb{F}_p} \xi(b) = 0,$$

by (3.2), thus giving us another $p-1$ orthogonal shift–frequency translates of z orthogonal to x . In total, there are $2(p+1)(p-1) = 2p^2 - 2$ such translates, thus concluding the proof. \square

4. SPARK DEFICIENT GABOR FRAMES OVER CYCLIC GROUPS

Here we revisit the cyclic case. As it is already proven by the author [13], almost all windows generate full spark Gabor frames, so the spark deficient Gabor frames are generated by exceptional vectors. When the order of the group is an odd, square-free integer, then all eigenvectors of certain unitaries belonging to the Clifford group generate spark deficient Gabor frames [4]. The motivation behind this result in [4] was to establish a connection between equiangularity of a Gabor frame (SIC-POVM existence) and full spark, if any. In 3 dimensions, the family of SIC-POVMs generated by vectors of the form $(0, 1, -e^{i\theta})$ is always spark deficient, and Lane Hughston [8] first established a connection between the linear dependencies that arise from this SIC-POVM for $\theta = 0$ or $2\pi/9$ and the inflection points of an elliptic curve. In general, it was proven in [4] that when N is an odd, square-free integer divisible by 3, all eigenvectors of the *Zauner unitary matrix* generate spark deficient Gabor frames. Zauner's conjecture [20] states that an eigenvector of this matrix generates a SIC-POVM, i. e. a maximal equiangular tight frame. If it is true, then for all odd, square-free dimensions, this equiangular tight frame is not full spark. This is another example that showcases the difference between a nice algebraic property of a Gabor frame (full spark) and a nice geometric one (equiangularity); it is not necessary that both of them can appear, even when the second one appears at all. For unit norm tight frames in general, this is further explained in [10]; see also [6, 9] where an infinite family of spark deficient equiangular tight frames is constructed, of arbitrarily high dimension.

When N is not divisible by 3, it is not known whether this SIC-POVM is also full spark or not. For example, it is full spark when $N = 8$ [4], being the first construction at that time of a full spark Gabor frame in 8 dimensions².

Concerning the eigenvectors of other Clifford unitaries, they also generate spark deficient Gabor frames as long as the (projective) order of the matrix divides N . We will extend the results of section 7 in [4], "Generalisation to other symplectic unitaries", to all odd dimensions N and unitaries whose order is not coprime to N .

Theorem 4.1. *Let N be an odd integer. Then, any eigenvector of the unitary U_F generates a spark deficient Gabor frame, where $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ and $\gcd(\text{ord}(F), N) > 1$.*

This is a direct consequence of the following theorem from [4], slightly rephrased in order to accommodate the terminology of this paper, with the simple observation that if $\text{ord}(F) = n$ and $\gcd(n, N) = d > 1$, then the eigenvectors of U_F are also eigenvectors of $e^{i\theta}U_F^{n/d} = e^{i\theta}U_{F^{n/d}}$ (the phase $e^{i\theta}$ is arbitrary), while $\text{ord}(F^{n/d}) = d > 1$, hence $\text{ord}(F^{n/d})$ divides N .

We call $\mathbf{x} \in (\mathbb{Z}/N\mathbb{Z})^2$ *F-full*, if the vectors $\mathbf{x}, F\mathbf{x}, \dots, F^{n-1}\mathbf{x}$ are all distinct, where $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ and $n = \text{ord}(F)$.

Theorem 4.2 (Theorem 5 [4], odd version). *Let N be an odd positive integer and $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$, and let $n = \text{ord}(F)$. Suppose*

- (1) $n > 1$.
- (2) n divides N .
- (3) $\text{Tr } U_F \neq 0$.
- (4) *There exist N distinct points in $(\mathbb{Z}/N\mathbb{Z})^2$ that are F -full.*

Then all eigenvectors of U_F generate spark deficient Gabor frames.

Conditions (3) and (4) always hold when N is odd, as the following two Lemmata show; this was proven in [4] for N odd square-free³.

Lemma 4.3. *Let N be an odd positive integer. Let $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ be arbitrary. Then the number of F -full points in $(\mathbb{Z}/N\mathbb{Z})^2$ is $\geq N\varphi(N)$, where φ is Euler's function.*

²Explicit construction of a full spark Gabor frame in every dimension was later shown by the author [13].

³See Lemmata 7 and 8 in [4]

Lemma 4.4. *Let N be an odd positive integer. Then $|\mathrm{Tr}(U_F)| \geq 1$ for all $F \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$.*

We will dedicate the rest of this section to the proofs of these two Lemmata. For basic facts about the field of p -adic numbers, \mathbb{Q}_p and its algebraic extensions, we refer the reader to [2, 14].

4.1. Proof of Lemma 4.3. Let $F \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ and $F_i \in \mathrm{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$ be the reduction of F modulo $p_i^{r_i}$, $1 \leq i \leq s$. Similarly, with $\mathbf{x} \in (\mathbb{Z}/N\mathbb{Z})^2$ and $\mathbf{x}_i \in \mathrm{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$. It is not hard to show that if each \mathbf{x}_i is F_i -full, then \mathbf{x} is F -full, a fact also shown in [4]. By multiplicativity of the Euler function, it suffices to consider $N = p^r$, a power of an odd prime.

The case $r = 1$ was treated in [4]. The technique was to find the Jordan canonical form of F , considering a quadratic extension of the field $\mathbb{Z}/p\mathbb{Z}$ if necessary (i. e. \mathbb{F}_{p^2}); then we can control the powers of F and can count the points in $(\mathbb{Z}/p\mathbb{Z})^2$ that are F -full.

When $r > 1$, $\mathbb{Z}/N\mathbb{Z}$ is no longer a field, so the Jordan canonical form does not always exist, but as we shall see below, in these exceptional cases, the order of F is equal to p^m or $2p^m$, for some $m \leq r$, so we only need to enumerate the points in $(\mathbb{Z}/N\mathbb{Z})^2$ that are fixed by $F^{p^{m-1}}$ or $F^{2p^{m-1}}$ accordingly, and as it turns out, this is an easy task.

It would be convenient to consider an arbitrary lift of the matrix

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to a matrix in $\tilde{F} \in \mathrm{SL}(2, \mathbb{Z}_p)$; since $F \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, then at least one of the entries a, b, c, d , is not divisible by p , say a . Then, lift a, b, c , arbitrarily, to $\tilde{a}, \tilde{b}, \tilde{c}$, and put $\tilde{d} = \tilde{a}^{-1}(1 + \tilde{b}\tilde{c})$. We also put $t = \mathrm{Tr}(F)$, $\tilde{t} = \mathrm{Tr}(\tilde{F})$, $\Delta = t^2 - 4$, $\tilde{\Delta} = \tilde{t}^2 - 4$, the discriminants of the characteristic polynomials of F, \tilde{F} , respectively. Finally, we put

$$\lambda = \frac{t + \sqrt{\Delta}}{2},$$

and the other root of the characteristic polynomial is λ^{-1} . We distinguish the following cases:

$p \nmid \Delta$ In this case, $\lambda \not\equiv \lambda^{-1} \pmod{p}$; otherwise, we would have

$$\Delta \equiv (\lambda + \lambda^{-1})^2 - 4 \equiv \lambda^2 + \lambda^{-2} - 2 \equiv 0 \pmod{p}.$$

We reduce the entries of $F \pmod{p}$. Since $\lambda \not\equiv \lambda^{-1} \pmod{p}$, F is diagonalizable in $\mathbb{Z}/p\mathbb{Z}$ when $(\frac{\Delta}{p}) = 1$ or in a quadratic extension, namely \mathbb{F}_{p^2} , when $(\frac{\Delta}{p}) = -1$. In both cases, we consider the field $K = \mathbb{Q}_p(\sqrt{\tilde{\Delta}})$, whose ring of integers is $\mathcal{O}_K = \mathbb{Z}_p[\sqrt{\tilde{\Delta}}]$ and the unique prime ideal is $p\mathcal{O}_K = p\mathbb{Z}_p[\sqrt{\tilde{\Delta}}]$. This extension is unramified, as $p \nmid \Delta$, hence the degree of the extension is equal to the degree of the extension of the residue fields. Therefore, the residue field of K is \mathbb{F}_p when $(\frac{\Delta}{p}) = 1$ and \mathbb{F}_{p^2} otherwise.

So, there is a nonsingular matrix X with entries in the residue field of K such that

$$(4.1) \quad FX \equiv X \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \pmod{p\mathcal{O}_K},$$

the congruence meaning that we consider each entry $\pmod{p\mathcal{O}_K}$. We can lift $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$ to a 2×2 matrix with entries in \mathcal{O}_K , such that (4.1) becomes an equality in \mathcal{O}_K (and holds \pmod{N} , in particular). Indeed, if b is not divisible by p , then we lift x, z arbitrarily, and then put $y = \tilde{b}^{-1}(\lambda - \tilde{a}x)$, $w = \tilde{b}^{-1}(\lambda - \tilde{a}z)$, and a similar lift is possible if c is not divisible by p . If both b and c are divisible by p , then $F \pmod{p}$ is diagonal, hence $F \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ or $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \pmod{p}$. Without loss of generality, we may assume that the first congruence holds. Lift x, w , arbitrarily, and then put $y = (\lambda - \lambda^{-1})^{-1}\tilde{c}x$, and $z = (\lambda^{-1} - \lambda)^{-1}\tilde{b}w$. We notice that since $p \nmid \det(X)$,

then $X^{-1} \in \text{GL}(2, \mathcal{O}_K)$; we conclude that in all cases where $p \nmid \Delta$, F is equivalent to a diagonal matrix, with entries perhaps in a larger ring. It is evident that in this case, the number of F -full points is $N^2 - 1$, since $p \nmid \lambda$, and $\lambda \not\equiv 1 \pmod{p}$.

$\boxed{p \mid \Delta}$ Reducing the matrix $F \pmod{p}$, we obtain a double eigenvalue, equal to ± 1 . Then, the Jordan canonical form of F is

$$\begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}$$

where $\beta = 0$ or $\beta = 1$. It is clear that $F^p \equiv \pm I \pmod{p}$ and $F^{2p} \equiv I \pmod{p}$, or $F^{2p} \equiv I + pA \pmod{p^2}$, for some matrix A . Raising both sides to the p -th power, we get $F^{2p^2} \equiv I + p^2A \pmod{p^3}$, and proceeding inductively we can show that

$$F^{2p^{r-1}} = I + \frac{N}{p}A,$$

hence $F^{2N} = I$. This shows that the order of F is either p^m or $2p^m$, for some $m \leq r$.

Suppose first that the order of F is p^m ($m \geq 1$); then, an element of $(\mathbb{Z}/N\mathbb{Z})^2$ is F -full, if and only if it is not fixed by $F^{p^{m-1}}$ (this follows from the fact that the cardinality of the orbit of any element under a group, divides the order of the group), and the latter is equivalent to the condition that this element is $F^{p^{m-1}}$ -full. Therefore, we can reduce to the case where $m = 1$, that is, the order of F is p . Since the number of F -full points is the same in the conjugacy class of F , we may further assume that F reduced \pmod{p} is equal to $\begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}$. Now, let k be the smallest positive integer for which we have

$$F \equiv \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} + p^{k-1}D \pmod{p^k}$$

for some matrix $D \not\equiv \mathbf{O} \pmod{p}$, where \mathbf{O} is the zero matrix. We have $2 \leq k \leq r+1$. If $\beta = 0$, then $k = r$; if $k < r$, then

$$F^p \equiv I + p^k D \pmod{p^{k+1}},$$

hence $F^p \neq I$, a contradiction. Similarly, if $k = r+1$, then $F = I$, which is also a contradiction.

So, $F = I + \frac{N}{p}D$. A vector $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2$ is fixed by F if and only if

$$D\mathbf{x} \equiv \mathbf{0} \pmod{p}.$$

The set of such vectors reduced \pmod{p} form a proper vector subspace of $(\mathbb{Z}/p\mathbb{Z})^2$, so they are at most p . Then, the number of all the possible lifts of these vectors is at most \pmod{N} is $p^{2(r-1)} \cdot p = p^{2r-1}$. Therefore, the number of F -full vectors in this case is at least $p^{2r} - p^{2r-1} = N\varphi(N)$.

If $\beta = 1$, then

$$F^p \equiv \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} + p^{k-1} \sum_{\kappa+\mu=p-1} \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \pmod{p^k}.$$

We put $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ and compute the above sum \pmod{p} :

$$\begin{aligned} \sum_{\kappa+\mu=p-1} \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} &= \sum_{\kappa+\mu=p-1} \begin{pmatrix} d_1 + \kappa d_3 & \mu d_1 + \kappa \mu d_3 + d_2 + \kappa d_4 \\ d_3 & \mu d_3 + d_4 \end{pmatrix} \\ &\equiv \mathbf{O} \pmod{p} \end{aligned}$$

since

$$\sum_{\kappa+\mu=p-1} 1 = p, \quad \sum_{\kappa+\mu=p-1} \kappa = \sum_{\kappa+\mu=p-1} \mu = p \cdot \frac{p-1}{2}, \quad \sum_{\kappa+\mu=p-1} \kappa \mu = p \cdot \left(\frac{(p-1)^2}{2} - \frac{(p-1)(2p-1)}{6} \right).$$

But then, $F^p \not\equiv I \pmod{p^k}$, a contradiction if $k \leq r$; if $k = r + 1$, then $F^p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \neq I$. We conclude that if the order of F is p and $r \geq 2$, then $\beta = 0$ (the case $\beta \neq 0$ can only occur when $r = 1$, but this was treated in [4]).

Next, suppose that the order of F is $2p^m$. Then, a vector is F -full if and only if it is not fixed by F^{p^m} or $F^{2p^{m-1}}$. But $F^{p^m} = -I$, which only fixes the zero vector, so we only need to exclude the vectors fixed by $F^{2p^{m-1}}$; however, this matrix has order p , so the above analysis applied to $F^{2p^{m-1}}$ yields the fact that the number of F -full points is at least $N\varphi(N)$.

4.2. Proof of Lemma 4.4. The trace $\text{Tr}(U_F)$ is a quadratic Gauss sum [1]; we will use the following lemma by Turaev [18] (see Lemma 1) which gives the absolute value of such a sum over an arbitrary finite abelian group G . Moreover, by (2.1) we may assume that N is a power of an odd prime, p .

Let's fix some notation first; $q : G \rightarrow \mathbb{Q}/\mathbb{Z}$ denotes an arbitrary quadratic form on the finite abelian group G . Such a function is a quadratic form if the expression $b^q(x, y) = q(x + y) - q(x) - q(y)$ is bilinear (*we do not require homogeneity*). The Gauss sum $\Gamma(G, q)$ is defined to be

$$\frac{1}{|G|^{1/2}} \sum_{x \in G} e^{2\pi i q(x)}.$$

Lastly, for easy reference to the explicit formulae for the unitary matrices U_F given in [1], we decided to use the bra-ket notation; the set of (column) vectors

$$|0\rangle, |1\rangle, \dots, |N-1\rangle,$$

is the standard basis of \mathbb{C}^N , and $\langle \varphi |$ is the conjugate transpose of $|\varphi\rangle$.

Lemma 4.5 (Lemma 1 [18]). *Let B be the kernel of the homomorphism $G \rightarrow \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ adjoint to the pairing b^q . If $q(B) \neq 0$, then $\Gamma(G, q) = 0$. If $q(B) = 0$, then $|\Gamma(G, q)| = |B|^{1/2}$.*

If $p \nmid b$, then the matrix F is called *prime*, and from the explicit formulae of [1] (see Lemma 2 and Lemma 4), we get

$$U_F = \frac{e^{i\theta}}{\sqrt{N}} \sum_{r,s=0}^{N-1} \tau^{b^{-1}(as^2 - 2rs + dr^2)} |r\rangle \langle s|,$$

where θ is an arbitrary phase, and b^{-1} the inverse of $b \pmod{N}$, hence

$$\text{Tr}(U_F) = \frac{e^{i\theta}}{\sqrt{N}} \sum_{r=0}^{N-1} \tau^{b^{-1}(t-2)r^2}.$$

where $\tau = -e^{\frac{\pi i}{N}}$ and $t = a + d = \text{Tr}(F)$. Putting $G = \mathbb{Z}/N\mathbb{Z}$ and

$$q(r) = \frac{b^{-1}(t-2)(N+1)}{2N} r^2,$$

we get $\text{Tr}(U_F) = e^{i\theta} \Gamma(G, q)$. q is a well-defined quadratic form on G ; indeed, as $r^2 \equiv r'^2 \pmod{2N}$, when $r \equiv r' \pmod{N}$, when N is odd. The associated bilinear pairing is

$$b^q(r, s) = \frac{b^{-1}(t-2)(N+1)}{N} rs,$$

and $r \in B$ if and only if $b^q(r, 1) = 0$, or equivalently, if

$$b^{-1}(t-2)r \equiv 0 \pmod{N}.$$

So, if $r \in B$ is arbitrary, then N divides $b^{-1}(t-2)r^2$, hence $2N$ divides $b^{-1}(t-2)(N+1)r^2$, which shows that $q(r) = 0$. This proves that $q(B) = 0$, hence $|\Gamma(G, q)| = |B|^{1/2} \geq 1$ and $|\text{Tr}(U_F)| \geq 1$.

Now, assume that $p \mid b$; then $p \nmid d$ (otherwise $\det(F)$ would be divisible by p) and we can write F as a product of two prime matrices, as follows:

$$F = F_1 F_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$$

and by Lemma 4 [1], we have $U_F = U_{F_1} U_{F_2}$, where

$$U_{F_1} = \frac{e^{i\theta_1}}{\sqrt{N}} \sum_{u,v=0}^{N-1} \tau^{2uv} |u\rangle \langle v|$$

and

$$U_{F_2} = \frac{e^{i\theta_2}}{\sqrt{N}} \sum_{v,w=0}^{N-1} \tau^{d^{-1}(cw^2-2vw-bv^2)} |v\rangle \langle w|,$$

where θ_1, θ_2 arbitrary phases, hence

$$U_F = \frac{e^{i\theta}}{N} \sum_{u,w=0}^{N-1} \sum_{v=0}^{N-1} \tau^{2uv+d^{-1}(cw^2-2vw-bv^2)} |u\rangle \langle w|$$

and

$$\text{Tr}(U_F) = \frac{e^{i\theta}}{N} \sum_{u,v=0}^{N-1} \tau^{cd^{-1}u^2+2(1-d^{-1})uv-bd^{-1}v^2},$$

where $\theta = \theta_1 + \theta_2$. So, if we put $G = (\mathbb{Z}/N\mathbb{Z})^2$ and $q : G \longrightarrow \mathbb{Q}/\mathbb{Z}$ the quadratic form

$$q(u, v) = \frac{N+1}{2N} (cd^{-1}u^2 + 2(1-d^{-1})uv - bd^{-1}v^2)$$

then $\text{Tr}(U_F) = e^{i\theta} \Gamma(G, q)$. The associated bilinear form is

$$b^q((u, v), (r, s)) = \frac{N+1}{N} (u \ v) A \begin{pmatrix} r \\ s \end{pmatrix}$$

where

$$A = \begin{pmatrix} cd^{-1} & 1-d^{-1} \\ 1-d^{-1} & -bd^{-1} \end{pmatrix}.$$

Now, let $(u \ v) \in B$ be arbitrary. Then,

$$(u \ v) A \equiv (0 \ 0) \pmod{N},$$

otherwise, we would have either $b^q((u, v), (1, 0)) \neq 0$ or $b^q((u, v), (0, 1)) \neq 0$. In particular, N divides $b^q((u, v), (u, v))$, and since N is odd, $2N$ divides

$$(N+1)(u \ v) A \begin{pmatrix} u \\ v \end{pmatrix}$$

which yields $q(u, v) = 0$. Thus, $q(B) = 0$, and $|\text{Tr}(U_F)| = |\Gamma(G, q)| = |B|^{1/2} \geq 1$.

5. UNCERTAINTY PRINCIPLES

The full spark property of (almost all) Gabor frames of windows defined over finite cyclic groups implies the following inequality for the short-time Fourier transform of f :

$$\|V_\varphi f\|_0 \geq N^2 - N + 1,$$

where N is the size of said group, for almost all $\varphi \in \mathbb{C}^N$ and all nonzero $f \in \mathbb{C}^N$ [11, 13, 15]. A possible connection between the set of pairs of the form $(\|f\|_0, \|\widehat{f}\|_0)$, denoted by F , and the set F_φ of all pairs of the form $(\|f\|_0, \|V_\varphi f\|_0 - N^2 + N)$ (for both sets we take f nonzero) was investigated in [11]. In particular, the following problem was proposed.

Problem 5.1 ([11]). *Is it true that $F = F_\varphi$ for almost all φ ?*

When $N = p$ a prime number, this problem was solved to the affirmative [11]. One has an exact characterization of the set F [17] and the fact that all minors of the Gabor synthesis matrix are nonzero for all φ except for a set of measure zero (Theorem 4 [12]), leads to a characterization of the set F_φ , and equality between F and F_φ is easily confirmed. When N is composite, however, there is no exact characterization for the set F , so it is more difficult to obtain equality; this was confirmed numerically for dimensions up to 6 [11]. The question is whether we can prove equality between those two sets without using the characterization of F . We will show that one inclusion is possible, but the other one, namely $F_\varphi \subseteq F$ seems much harder to prove, if true.

As a final remark, we note that the spark deficiency of all Gabor frames of windows defined over abelian, non-cyclic groups, implies that equality between F and F_φ can never be achieved, simply because there are $f \in \mathbb{C}^G$ for which $\|V_\varphi f\|_0 \leq N^2 - N$, as shown in the proof of Theorem 3.3.

A useful identity is the following:

$$(5.1) \quad \|V_\varphi f\|_0 = \sum_{j=0}^{N-1} \left\| \widehat{T^j \varphi} \cdot f \right\|_0.$$

Theorem 5.2. *For almost all φ the inclusion $F \subseteq F_\varphi$ holds. In addition, this φ can be taken to generate a full spark Gabor frame.*

Proof. First, we may restrict our attention to φ generating a full spark Gabor frame, as we already know that almost all φ satisfy this condition. This implies that all coordinates of φ are nonzero, otherwise the frequency translates of φ would form a singular matrix. Next, for any pair $(k, l) \in F$ we consider $f_{k,l} \in \mathbb{C}^N$ with $\|f_{k,l}\|_0 = k$ and $\|\widehat{f_{k,l}}\|_0 = l$. We may rewrite (5.1) as

$$(5.2) \quad \left\| V_\varphi \frac{f_{k,l}}{\varphi} \right\|_0 = \sum_{j=0}^{N-1} \left\| \frac{\widehat{T^j \varphi}}{\varphi} \cdot f_{k,l} \right\|_0 = \left\| \widehat{f_{k,l}} \right\|_0 + \sum_{j=1}^{N-1} \left\| \frac{\widehat{T^j \varphi}}{\varphi} \cdot f_{k,l} \right\|_0 = l + \sum_{j=1}^{N-1} \left\| \frac{\widehat{T^j \varphi}}{\varphi} \cdot f_{k,l} \right\|_0.$$

It suffices to show that almost all φ satisfy

$$\left\| \frac{\widehat{T^j \varphi}}{\varphi} \cdot f_{k,l} \right\|_0 = N,$$

for all $(k, l) \in F$ and $1 \leq j \leq N-1$, or equivalently, it suffices to show that

$$\Phi \sum_{g=0}^{N-1} \xi(g) f_{k,l}(g) \frac{\varphi(g-j)}{\varphi(g)} \neq 0,$$

for almost all $\varphi \in \mathbb{C}^N$, all characters ξ , $(k, l) \in F$, $1 \leq j \leq N-1$, where Φ is the product of the coordinates of φ . But the left-hand side is a polynomial in the coordinates of φ with coefficients of the form $\xi(g) f_{k,l}(g)$, which shows that every such polynomial is nonzero, as the

functions $f_{k,l}$ are not identically zero. Therefore, φ has to avoid the zero set of finitely many nonzero polynomials, whose union is of measure zero. Thus, almost all φ satisfy

$$\left\| V_{\varphi} \frac{f_{k,l}}{\varphi} \right\|_0 = N^2 - N + l,$$

for every $(k, l) \in F$, as desired. \square

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